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# ON COMPOUND APPROXIMATIONS

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*Abstract.* We prove that Le Cam's theorem for compound Poisson can be extended to a much larger set of compound distributions. Signed compound negative binomial measure is introduced. Relations between compound Poisson and other compound approximations are discussed.

**1. Introduction.** Poisson distribution was studied in a comprehensive way by many authors, see [1], [5], [9]. Compound Poisson, though not so completely explored, has also received a lot of attention, see, for example, the classical papers by Le Cam [17], [18]. Interest in compound Poisson is especially fast growing in recent years – see [4], [7], [8], [20], [23] and references therein. Remarkably it turned out that compound Poisson can be of major practical importance, see [10], [11], [14]. In comparison with compound Poisson very little general results were obtained for other compound distributions. This paper deals with theoretical aspects of general compound approximations. We prove that the accuracy of compound approximation depends only on its simplest characteristics. It does not really matter is the structure of compound distribution of the exponential form or not. It is possible to extend results originally obtained for compound Poisson to a much larger set of compound distributions. Mainly we concentrate on the extension of Le Cam's [18] theorem. Le Cam's result is one of the most general results ever obtained for compound Poisson and assumptions in it are minimal: only the independence of summands. Thus, one may suspect that the estimate depends heavily on the exponential structure of approximation. As we prove in this paper, Le Cam's result depends on the exponential structure only in this way: it allows us to reduce the decomposition of approximated distribution from the approximating distribution. Below we review Le Cam's [18] theorem, but first some notation is needed.

Let $\mathscr{F}$ be the set of all distributions. Let $E_a$ denote the distribution concentrated at a point $a$, $E \equiv E_0$. The notation $C_m$ is used to denote positive absolute constants. Products and powers of measures are understood in the convolution sense: $FG = F * G$, $F^n = F^{*n}$, $F^0 = E$. For any finite variation measure $W$, we denote its Fourier–Stieltjes transform by $\hat{W}(t)$, and its exponential measure by

$$\exp\{W\} = \sum_{k=0}^{\infty} W^k/k!.$$

The analogue of the uniform Kolmogorov distance will be denoted by

$$|W| = \sup_x |W\{(-\infty, x)\}|$$

and the total variation (the norm) of $W$ by

$$\|W\| = W^+\{\mathbf{R}\} + W^-\{\mathbf{R}\}.$$

Note that the total variation has the following useful properties: for any $F \in \mathcal{F}$

$$\|FW\| \leqslant \|F\| \, \|W\|, \qquad \|F\| = 1, \qquad \|W\|/2 \leqslant \sup |W\{B\}| \leqslant \|W\|,$$

where the supremum is taken over all Borel sets. For $F \in \mathcal{F}$, $h \geqslant 0$, Levy's concentration function is defined by

$$Q(F, h) = \sup_x F\{[x, x+h]\}.$$

The real part of $\hat{F}(t)$ is denoted by $\operatorname{Re}\hat{F}(t)$. We define the compound distribution by

$$\varphi(F) = \sum_{i=0}^{\infty} p_i F^i, \quad 0 \leqslant p_i \leqslant 1, \quad \sum_{i=0}^{\infty} p_i = 1. \tag{1.1}$$

It is easy to check that $\hat{\varphi}(F)(t) = \varphi(\hat{F}(t))$.

Remark. Note that (1.1) can be viewed as a distribution of a random number of summands, i.e., let $\xi_1, \xi_2, \ldots$ be independent identically distributed random variables with common distribution $F$, and let $\zeta$ be a nonnegative integer-valued random variable independent of $\{\xi_i\}_{i=1}^{\infty}$. Then $\varphi(F)$ is the distribution of $\sum_{j=1}^{\zeta} \xi_j$.

Remark. Note also that $F$ can be regarded as the degenerate case of compound distribution, i.e., $F = \varphi(F)$ with $\varphi(E_1) \equiv E_1$.

In 1965 Le Cam obtained the following result:

THEOREM 1.1 (Le Cam [18]). *The following inequality holds:*

$$\sup_{F \in \mathcal{F}} \inf_{\alpha} |F^n - E_{-n\alpha} \exp\{n(FE_\alpha - E)\}| \leqslant C_1 n^{-1/3}. \tag{1.2}$$

Le Cam's result is closely related to the first uniform Kolmogorov's theorem, i.e., to the problem of approximation of arbitrary $F^n$ by the class of all infinitely divisible distributions. This problem was considered by Kolmogorov, Prohorov, Ibragimov and Presman, Meshalkin and many others. Though the optimal rate was determined by Arak [2] to be of order $n^{-2/3}$, Arak's approximation had no such explicit form as compound Poisson in (1.2). (For a comprehensive discussion on the problem we refer readers to the book by Arak and Zaĭtsev [3].) There are some remarks on (1.2):

1. the estimate in (1.2) is uniform with respect to $\mathscr{F}$, i.e., an absolute constant $C_1$ is the same for *all* distributions;

2. from (1.2) we see that the limiting distribution is not necessarily the best choice for approximation (limiting distribution may not even exist but (1.2) holds);

3. note that (1.2) can be rewritten as

$$\sup_{F\in\mathscr{F}}\inf_{\alpha}|(FE_\alpha)^n-\exp\{n(FE_\alpha-E)\}|\leqslant C_1 n^{-1/3}.$$

Hence, it is crucial to center $F$ *before* applying the compound Poisson approximation. This fact is a very important one, because the usual choice of compound Poisson does not include the centering of distribution, see, for example, [4]. As follows from (1.2), it is sometimes necessary to center the distribution loosing even such a nice property as latticeness, but gaining in accuracy. The estimate in (1.2) is optimal (cf. [15]); for the proof see [3]. Note that for symmetric distributions (1.2) can be improved and has already been generalized by Zaĭtsev [25]. We extend Le Cam's result in Section 2.

The case of compound negative binomial distribution is considered in Section 3. Quite recently it was proved that in many situations signed compound Poisson measures can dramatically improve the rate of approximation, see [14], [16]. The improvement is possible even in the general case, as considered in (1.2), see [6]. We introduce the signed compound negative binomial measure and prove that it has an analogous property.

In Section 4 we show how the solution of Urbanik's problem enables us to review the role of the Poisson distribution in the classical theory of infinitely divisible distributions.

**2. Extension of Le Cam's theorem.** First we introduce the set of lattice distributions that will serve as a basis for compound distributions. Let $\varphi(F)$ be defined as in (1.1). Then $\varphi(E_1)$ is a lattice distribution concentrated on non-negative integers with probabilities $\varphi(E_1)\{k\}=p_k$. We denote the factorial moments of $\varphi(E_1)$ by

$$\nu_j(\varphi)=\sum_{k=0}^{\infty}k(k-1)\ldots(k-j+1)p_k,\quad j=1,2,\ldots$$

Let us denote by $\mathscr{M}(a,b)$ the class of lattice distributions on $\{0,1,2,\ldots\}$, having $\nu_1(\varphi)=a$, $\nu_2(\varphi)\leqslant b$. Quite analogously to the well-known expansion in factorial moments we can show that for $\varphi(E_1)\in\mathscr{M}(a,b)$ and any $F\in\mathscr{F}$ the following relation holds true:

$$\varphi\big(\hat{F}(t)\big)=1+a\big(\hat{F}(t)-1\big)+\theta|\hat{F}(t)-1|^2/2\quad(t\in\boldsymbol{R}). \tag{2.1}$$

Here $\theta\leqslant\nu_2(\varphi)\leqslant b$, $t\in\boldsymbol{R}$.

In the following we need the decomposition of $F$. Let $p=n^{-1/3}$, $F\in\mathscr{F}$. Then there exist $a\in\boldsymbol{R}$, $h_-$, $h_+>0$, $A$, $B\in\mathscr{F}$ such that

$$FE_a=(1-p)A+pB, \tag{2.2}$$

$$A\{[-h_-, h_+]\}=B\{(-\infty, -h_-]\cup[h_+, \infty)\}=1, \tag{2.3}$$

$$\int xA\{dx\}=0, \tag{2.4}$$

$$FE_a\{(-\infty, -h_-]\}\geqslant p/2, \quad FE_a\{[h_+, \infty)\}\geqslant p/2. \tag{2.5}$$

For the proof see [15], [18].

Decomposition (2.2) was used by Kolmogorov and Prohorov, who applied the Gaussian approximation to $A$. Le Cam's result is remarkable also in the sense that it was probably one of the first general results where the approximating measure had no Gaussian component. On the contrary, even for the most smooth $F$ the compound Poisson has an atom at zero. The Kolmogorov –Prohorov approach and Le Cam's approach were combined by Zaĭtsev [24].

Now we give the main result of this paper.

THEOREM 2.1. *Let $F\in\mathscr{F}$, $p=n^{-1/3}$ and $a$, $A$, $B$ be defined by* (2.2)–(2.5). *Then for any $\varphi_1(E_1)$, $\varphi_2(E_1)\in\mathscr{M}(1, C_2)$ the following inequality holds*:

$$|F^n-E_{-na}\varphi_1^n((1-p)A+pE)\varphi_2^n((1-p)E+pB)|\leqslant C_3 n^{-1/3}. \tag{2.6}$$

Remark. To get (1.2) one must take $\varphi_1(E_1)\equiv\varphi_2(E_1)\equiv\exp\{E_1-E\}$ and note that

$$\exp\{(1-p)(A-E)\}\exp\{p(B-E)\} \tag{2.7}$$

$$=\exp\{((1-p)A+pB)-E\}=\exp\{FE_a-E\}.$$

Proof of Theorem 2.1. It suffices to prove (2.6) for $n\geqslant 8^3(C_2+1)^3$. Assume first that $h=\max(h_-, h_+)>0$. Set

$$H_1=(1-p)A+pE, \quad H_2=(1-p)E+pB.$$

Then

$$Q((E_aF)^n, h)\leqslant C_4(np)^{-1/2}, \tag{2.8}$$

$$|(FE_a)^n-H_1^nH_2^n|\leqslant C_5(np)^{-1/2}+C_6(p/n)^{1/2}.$$

(See, for example, [15].)

By the variant of Esseen's smoothing lemma (see [18]), we have

$$|H_1^nH_2^n-\varphi_1^n(H_1)H_2^n| \tag{2.9}$$

$$\leqslant C_7\int_0^{1/(2h+2hC_2)}|\hat{H}_1^n(t)-\varphi_1^n(\hat{H}_1(t))|\frac{dt}{t}+C_8 Q(H_1^nH_2^n, 2h(C_2+1)).$$

From (2.8) it follows that

$$(2.10)\quad Q(H_1^n H_2^n, 2h(C_2+1)) \leqslant C_9 Q(H_1^n H_2^n, h) \leqslant C_9(2|H_1^n H_2^n-(FE_a)^n|+Q((FE_a)^n, h)) \leqslant C_{10}((np)^{-1/2}+(p/n)^{1/2}).$$

From (2.1), (2.3), (2.4) we infer that, for $|t| \leqslant 1/(2h+2hC_2)$,

$$(2.11)\quad \begin{aligned}|\varphi_1(\hat{H}_1(t))| &\leqslant |1+(\hat{H}_1(t)-1)|+C_2|\hat{H}_1(t)-1|^2/2 \leqslant |1-\sigma^2(1-p)t^2/2| \\ &\quad +C_2(1-p)^2(\sigma^2 t^2)^2/8+\sigma^2 h|t|^3/6 \\ &\leqslant 1-\sigma^2 t^2/4(1-C_2(th)^2/2-2|th|/3) \leqslant 1-\sigma^2 t^2/8 \\ &\leqslant \exp\{-\sigma^2 t^2/8\},\end{aligned}$$

where $\sigma^2 = \int x^2 A\{dx\}$.

Analogously, for $|t| \leqslant 1/(2h+2hC_2)$

$$(2.12)\quad |\hat{H}_1(t)| \leqslant \exp\{-\sigma^2 t^2/8\}.$$

Consequently, from (2.1), (2.11), (2.12), for $|t| \leqslant 1/(2h+2hC_2)$, we obtain

$$(2.13)\quad \begin{aligned}|\varphi_1^n(\hat{H}_1(t))-\hat{H}_1^n(t)| &\leqslant ne^{1/8}\exp\{-n\sigma^2 t^2/8\}C_2|\hat{H}_1(t)-1|^2/2 \\ &\leqslant C_{11}n\exp\{-n\sigma^2 t^2/8\}(\sigma^2 t^2)^2 \leqslant C_{12}\exp\{-n\sigma^2 t^2/16\}\sigma|t|n^{-1/2}.\end{aligned}$$

Taking into account (2.13) and (2.10) we see that the right-hand side of (2.9) is majorized by $C_{13}((np)^{-1/2}+(p/n)^{1/2}+1/n)$.

It is easy to check that for any compound signed measure $\sum q_i F^i$ the following inequality holds:

$$(2.14)\quad \Big|\sum_{i=0}^{\infty} q_i F^i\Big| \leqslant \sum_{i=0}^{\infty} |q_i| = \Big\|\sum_{i=0}^{\infty} q_i E_i\Big\| = \Big\|\sum_{i=0}^{\infty} q_i E_1^i\Big\|.$$

Thus

$$(2.15)\quad \begin{aligned}|\varphi_1^n(H_1)H_2^n-\varphi_1^n(H_1)\varphi_2^n(H_2)| &\leqslant \|\varphi_1^n(H_1)\|\,|H_2^n-\varphi_2^n(H_2)| \\ &\leqslant \|((1-p)E+pE_1)^n-\varphi_2^n((1-p)E+pE_1)\|.\end{aligned}$$

We have

$$\varphi_2(1-p+pe^{it}) = 1+p(e^{it}-1)+C_2\theta p^2|e^{it}-1|^2/2$$

and

$$(2.16)\quad |\varphi_2(1-p+pe^{it})| \leqslant 1-p\sin^2\frac{t}{2} \leqslant \exp\Big\{-p\sin^2\frac{t}{2}\Big\}.$$

Now, we can proceed exactly as in the proof of Theorem 1 from [21] (see also [19]) proving that the right-hand side of (2.15) is inferior to $C_{14}n^{-1/3}$. This evidently completes the proof for $h > 0$. But in the case $h = 0$ the proof simply reduces to the estimate of (2.15). Thus the theorem is proved.

**3. Compound negative binomial measures.** For $F \in \mathscr{F}$ let us introduce the compound negative binomial distribution using the following notation:

$$(2E-F)^{-1} = \sum_{j=0}^{\infty} \left(\frac{1}{2}\right)^{j+1} F^j.$$

For the sake of convenience its $n$-fold convolution will be denoted by $(2E-F)^{-n}$. Note that the compound negative binomial distribution is infinitely divisible.

Obviously, in (2.6) we can take $\varphi_1(E_1) = \varphi_2(E_1) = (2E-E_1)^{-1}$. However, we cannot write the analogue of (2.7) for compound negative binomial. Nevertheless, there are situations when decomposition (2.2) can be reduced from approximating distribution. We do not know if it is possible in the general case, but we prove that it is possible at least for compound negative binomial.

THEOREM 3.1. *The following inequality holds*:

$$\sup_{F\in\mathscr{F}} \inf_{\alpha} |F^n - E_{-n\alpha}(2E-E_a F)^{-n}| \leqslant C_{15} n^{-1/3}. \tag{3.1}$$

Proof. We decompose $FE_a$ as in (2.2). Obviously, it suffices to consider the case $h > 0$, $n > 16^3$. Set $q = 1-p$.

In view of (2.6) it is sufficient to estimate

$$J = \left(2E-(qA+pE)\right)^{-n}\left(2E-(qE+pB)\right)^{-n} - (2E-FE_a)^{-n}.$$

But, by the smoothing inequality,

$$|J| \leqslant C_7 \int_0^{1/6h} |\hat{J}(t)| \frac{dt}{t} + C_8 Q\left(\left(2E-(qA+pE)\right)^{-n}\left(2E-(qE+pB)\right)^{-n}, 6h\right). \tag{3.2}$$

Just as in (2.10) we establish that the concentration function in (3.2) is inferior to $C_{16} n^{-1/3}$. Analogously to the proof of Theorem 2.1 we obtain

$$\left|1-q\left(\hat{A}(t)-1\right)\right|^{-1} \leqslant \exp\{-\sigma^2 t^2/8\} \quad \text{for } |t| \leqslant 1/6h.$$

On the other hand,

$$\begin{aligned} \left|\left(1-p\left(\hat{B}(t)-1\right)\right)^{-1}\right| &\leqslant \left|1+p\left(\hat{B}(t)-1\right)\right| + p^2 |\hat{B}(t)-1|^2 \\ &\leqslant 1+pq\left(\operatorname{Re}\hat{B}(t)-1\right) + 2p^2 |\operatorname{Re}\hat{B}(t)-1| \\ &\leqslant 1+p\left(\operatorname{Re}\hat{B}(t)-1\right)/2 \leqslant \exp\{p\left(\operatorname{Re}\hat{B}(t)-1\right)/2\}. \end{aligned} \tag{3.3}$$

Noting that for $|t| \leqslant 1/6h$ the estimate $1-\operatorname{Re}\hat{A}(t) \geqslant \sigma^2 t^2/3$ holds (see [3, Theorem 1.1.10]), we obtain

$$\begin{aligned} \left|\left(2-e^{ita}\hat{F}(t)\right)^{-1}\right| &\leqslant \left(2-\operatorname{Re}\left(e^{ita}\hat{F}(t)\right)\right)^{-1} \leqslant \left(1+\sigma^2 t^2 q/3 + p\left(1-\operatorname{Re}\hat{B}(t)\right)\right)^{-1} \\ &\leqslant \exp\{-\sigma^2 t^2/10 + p\left(\operatorname{Re}\hat{B}(t)-1\right)/3\}. \end{aligned} \tag{3.4}$$

We have

$$(3.5)\qquad \left|\left(1-q(\hat A(t)-1)\right)^{-1}\left(1-p(\hat B(t)-1)\right)^{-1}-\left(1-q(\hat A(t)-1)-p(\hat B(t)-1)\right)^{-1}\right| \leqslant \left|q(\hat A(t)-1)p(\hat B(t)-1)\right|.$$

But

$$(3.6)\qquad |\hat A(t)-1| \leqslant \sigma^2 t^2/2, \quad |\hat B(t)-1| \leqslant 2\sqrt{1-\operatorname{Re}\hat B(t)}.$$

Consequently, for $|t| \leqslant 1/6h$

$$|\hat J(t)| \leqslant C_{17}\exp\left\{-C_{18}n\left(\sigma^2 t^2+p(\operatorname{Re}\hat B(t)-1)\right)\right\}p\left(1-\operatorname{Re}\hat B(t)\right)^{1/2}\sigma^2 t^2 n.$$

From the last estimate and (3.2) it is not difficult to obtain

$$|J| \leqslant C_{19} n^{-1/3}.$$

Thus the theorem is proved.

It must be noted that the centering of $F$ does not depend on $\varphi$. Therefore, by the triangle inequality, we obtain the following corollaries:

COROLLARY 3.1. *Let* $\varphi_1(E_1)\in\mathscr{M}(1, C_{20})$, $\varphi_2(E_1)\in\mathscr{M}(1, C_{21})$. *Then*

$$(3.7)\qquad \sup_{F\in\mathscr{F}}\inf_{a}|\varphi_1^n(E_a F)-\varphi_2^n(E_a F)| \leqslant C_{22} n^{-1/3}.$$

COROLLARY 3.2. *The following inequality holds*:

$$(3.8)\qquad \sup_{F\in\mathscr{F}}\inf_{a}|(2E-E_a F)^{-n}-\exp\{n(E_a F-E)\}| \leqslant C_{23} n^{-1/3}.$$

Presman [19], Hipp [14], Kruopis [16], Čekanavičius [6] considered signed compound Poisson measures of the form $\exp\{\lambda(F-E)\}$ with $F\in\mathscr{F}$ and $\lambda\in\boldsymbol{R}$ (not necessarily nonnegative). Such measures can significantly improve the accuracy of approximation.

The idea to use signed compound measures probably was introduced by Le Cam [17], where the signed compound binomial measure was applied. In Le Cam's paper [18] it was proved that compound binomial distribution can be expressed as convolution of compound Poisson distribution with signed compound Poisson measure. However, compounding distributions had a much more complicated structure than the compound binomial. Thus, there is a very serious (and in general not solved) problem of what restrictions must be imposed on signed compound approximations. The usual practice is to use sign measures as some sort of asymptotic expansion to the main compound approximation, ensuring that their structures are quite comparable with the main term. We shall end this section by introducing signed compound negative binomial measure. For $F\in\mathscr{F}$, $\|F-E\| \leqslant 1/4$, let $\psi(F)$ denote the compound measure with Fourier–Stieltjes transform

$$\hat\psi(F)(t) = \psi(\hat F(t)) = \left(1-(\hat F(t)-1)+(\hat F(t)-1)^2\right)^{-1}.$$

Then the following result holds:

THEOREM 3.2. *Let $F \in \mathscr{F}$, $p = 1/10$, $a$, $A$, $B$ be defined by* (2.2)–(2.5). *Then*

$$|F^n - E_{-na}(2E-(qA+pE))^{-n}\psi^n(qE+pB)| \leqslant C_{24}n^{-1/2}. \tag{3.9}$$

Remark. Note that (3.9) provides the accuracy of order $n^{-1/2}$ for any $F$ and, unlike Berry–Esseen's theorem, there are no assumptions of the existence of moments.

Remark. Conditions on $F$ and $p$ in the definition of $\psi$ can be changed. We simply assumed conditions sufficient for finiteness of $\psi(F)$ variation.

Proof of Theorem 3.2. In [6] it was proved that for $p \equiv \text{const}$ the right-hand side of (2.6) is inferior to $C_{25}n^{-1/2}$. On the other hand, (2.7)–(2.13) hold true for $p \leqslant 1/10$. Therefore it suffices to prove that

$$\|(qE+pE_1)^n - \psi^n(qE+pE_1)\| \leqslant C_{26}n^{-1/2}.$$

But this estimate follows from the general result (Theorem 1) in [21]. Thus the theorem is proved.

**4. Compound distributions and the weak convergence of convolutions.** We considered so far Le Cam's theorem and its extensions. Of course, there are many other results that can be extended from compound Poisson to the general compound case. However, we have some doubts about this. Poisson and compound Poisson are probably the most convenient to use. The scope of this paper is not to discredit compound Poisson; on the contrary, we are very much in favour of it. Our main goal is to change the common-spread attitude toward Poisson and compound Poisson as to the distributions of utmost theoretical importance and uniqueness. We hope that we succeeded in proving that, as good as it is, compound Poisson is not unique and usually it suffices to know very little about the compound structure of distribution to get the meaningful results. Note also that all results above are formulated for convolutions. It would be interesting to have an answer to the problem of the possible extensions (or analogues) of Le Cam's theorem for some generalized convolutions that correspond to other schemes different from summation.

In this section we show that even in the most classical situation of characterization of infinitely divisible distributions the Poisson distribution is just one of many possibilities.

Let $\{X_{nk}\}$ $(n \geqslant 1,\ 1 \leqslant k \leqslant k_n)$ be the triangular array of independent random variables satisfying

$$\lim_{n\to\infty} \max_k P(|X_{nk}| > \varepsilon) = 0 \quad (\varepsilon > 0). \tag{4.1}$$

Then the set of possible limit distributions for the sum

$$\sum_{k=1}^{k_n} X_{nk} - A_n, \tag{4.2}$$

where $A_n$ is a suitable constant, coincides with the set of infinitely divisible distributions. The role of the Poisson distribution illustrates the following theorem, see [12, p. 74]:

THEOREM 4.1. *The totality of infinitely divisible distribution laws coincides with the totality of laws which are composed of a finite number of Poisson laws and of limits of these laws in the sense of weak convergence.*

We show that even in this situation Poisson distributions can be replaced by some others. This fact is closely connected with the solution of Urbanik's problem obtained by Hildebrand [13]. Urbanik [22] posed a problem on limiting behaviour of the sums of two-valued random variables. We do not review the problem and its solution in full (readers can find it in [13]), because for our purposes more important is the following adjacent result from [13]:

THEOREM 4.2 (Hildebrand [13]). *Let G be an infinitely divisible distribution. Let $p_{nk}$, $n \geqslant 1$, $1 \leqslant k \leqslant k_n$, be real numbers satisfying*

$$0 < p_{nk} < 1, \qquad \sum_{k:p_{nk} \leqslant \varepsilon} p_{nk} \to \infty \qquad (n \to \infty \textit{ for every } \varepsilon). \tag{4.3}$$

*Then there exist real numbers $a_{nk}$ such that if $X_{nk}$ are independent random variables with distributions*

$$P(X_{nk} = a_{nk}) = p_{nk}, \qquad P(X_{nk} = 0) = 1 - p_{nk}, \tag{4.4}$$

*then* (4.1) *holds and, with suitable constants $A_n$, the distribution of the sums* (4.2) *converges weakly to G.*

From Theorem 4.2 it is not difficult to deduce the following result:

THEOREM 4.3. *Let G be an infinitely divisible distribution. Let $p_{nk}$, $n \geqslant 0$, $1 \leqslant k \leqslant k_n$, be real numbers satisfying* (4.3) *and let*

$$p = \max_{1 \leqslant k \leqslant k_n} p_{nk} \to 0 \qquad (\textit{as } n \to \infty). \tag{4.5}$$

*Then there exist real numbers $a_{nk}$ $(k = 1, 2, \ldots, k_n)$ and constants $A_n$ such that for $\varphi_1(E_1), \ldots, \varphi_{k_n}(E_1) \in \mathcal{M}(1, C_{27})$ the convolution*

$$\sum_{k=1}^{k_n} \varphi_k\big((1-p_{nk})E + p_{nk} E_{a_k}\big) E_{-A_n} \tag{4.6}$$

*converges weakly to G as $n \to \infty$.*

Proof. From Theorem 4.2 we obtain $a_k$, $A_k$ such that

$$\prod_{k=1}^{k_n} \big((1-p_{nk} E + p_{nk} E_1\big) E_{-A_n}$$

converges weakly to $G$. Set for the sake of brevity

$$B_k = (1-p_{nk})E + p_{nk} E_{a_{nk}}, \qquad x_k = \sin(a_{nk} t/2).$$

Then

$$(4.7)\quad \Big|\prod_{k=1}^{k_n} \hat{B}_k(t)-\prod_{k=1}^{k_n} \varphi_k(\hat{B}_k(t))\Big|$$

$$\leqslant \sum_{k=1}^{k_n} \Big|\prod_{m=1}^{k-1} \hat{B}_m(t) \prod_{m=k+1}^{k_n} \varphi(\hat{B}_m(t))\Big| \big|\hat{B}_k(t)-\varphi_k(\hat{B}_k(t))\big|.$$

Just as in (2.16), for large $n$ we get

$$(4.8)\qquad |\hat{B}_m(t)|,\ |\varphi_m(\hat{B}_m(t))| \leqslant \exp\{-p_{nk}x_k^2\}.$$

On the other hand,

$$(4.9)\qquad \big|\hat{B}_k(t)-\varphi(\hat{B}_k(t))\big| \leqslant C_{27}|\hat{B}_k(t)-1|^2 \leqslant 4C_{27}p_{nk}px_k^2.$$

Applying (4.8) and (4.9) to (4.7) we infer that the left-hand side of (4.7) is inferior to

$$C_{28}\sum_{k=1}^{k_n} p_{nk}x_n^2 \exp\{-\sum_{j=1}^{k_n} p_{nj}x_j^2\} \leqslant C_{28}\,p.$$

The assertion of the theorem now follows from (4.5).

Remark. According to Hildebrand [13, p. 74] condition (4.5) is not very restrictive.

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